# ГИПОТЕЗА БИЛЯ И ВЕЛИКАЯ ТЕОРЕМА ФЕРМА КАК ОБРАТНЫЕ ЗАДАЧИ МАТЕМАТИЧЕСКОЙ ПСИХОЛОГИИ* 

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# BEAL'S CONJECTURE AND FERMAT'S LAST THEOREM AS REVERSE PROBLEMS OF MATHEMATICAL PSYCHOLOGY* 

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#### Abstract

АННОТАЦИЯ

Гипотеза Биля как обобщенная Великая теорема Ферма рассматривается с точки зрения математической психологии. Исследование гипотезы Биля проводится на базе геометрической теоремы Евклида и «чудесного доказательства» утверждения Ферма, записанного им на полях «Арифметики» Диофанта. Предлагается математическая реконструкция нативного доказательства Великой теоремы Ферма и решение проблемы Биля на ее основе. Гипотеза Биля и Великая теорема Ферма классифицируются в научном исследовании как обратные задачи математической психологии.

Ключевые слова: гипотеза Биля; Великая теорема Ферма; древнегреческая математика; математическая психология


#### Abstract

Beal's Conjecture as generalized Fermat's Last Theorem is considered from viewpoint of mathematical psychology. Investigation of Beal's Conjecture is conducted on the basis of geometrical Euclid's theorem and Fermat's "miraculous demonstration" of his proposition made by him on the margin of Diophantus' "Arithmetic". The mathematical reconstruction of native proof of Fermat's Last Theorem and solution of Beal's Conjecture on its base are suggested. The final status of both Beal's Conjecture and Fermat's Last Theorem is determined as reverse problems in mathematical psychology.


Keywords: Beal's Conjecture; Fermat's Last Theorem; ancient Greek mathematics; mathematical psychology

## I. Introduction. Historical roots of Beal's Conjecture and Fermat's Last Theorem.

Andrew Beal, a number theory enthusiast, formulated a conjecture generalizing Fermat's Last Theorem [4]. Apparently he was not content himself with geometric solution of the last issued in 1995 by A. Wiles [5] and offered a prize for solution of his generalizing conjecture [6] in order to inspire young people to research into Fermat's mathematics, believing that Fermat possessed a relatively simple arithmetic proof for his enigmatic proposition $[4 ; 6]$.

The Beal problem was also named the Beal Prize Conjecture by AMS [6] and the Beal, Granville, Tijdeman-Zagier Conjecture (Wikipedia, the free encyclopedia). Among well-known mathematics conjectures Beal's Conjecture is occupying a peculiar place being an announced generalization of Fermat's Last Theorem [4]. Both problems may be related to the part of number theory defined as arithmetic algebraic geometry including pure arithmetic methods of research.

There is enough evidence confirming the skill of ancient mathematicians to solve some algebraic equations with only arithmetic methods. These methods can enter into the sphere of arithmetic algebraic geometry and may be called arithmetic geometry methods. Corner-stones of arithmetic geometry of ancients were Euclid's theorem about proportional (geometric) means, unmeritedly forgotten in contemporary Diophantine geometry, and the Pythagorean theorem emerging from it. These theorems were widely known in ancient Greece and countries of the Great Silk Path. Further on it will be shown how they help to find a way to solve Beal's Conjecture and build modern reconstruction of possible Fermat's demonstration.

## II. Lemma: Fermat's Last Theorem. Solution of Fermat's Last Theorem as a reverse problem of mathematical psychology.

In fact, Fermat's Last Theorem is prelude for Beal's Conjecture. When proved it becomes a constitutional part of Beal's Conjecture solution. So we have the next Lemma: the following equation (1) with integers $z, x, y$ and natural exponent $n>2$ has no solution:

$$
\begin{equation*}
x^{n}+y^{n}=z^{n} \tag{1}
\end{equation*}
$$

Let us check this assertion. Suppose however that at least one solution was found. Then we shall try to construct such a solution and make certain of its impossibility.

Proof. In the beginning let us apply extension of the set of whole numbers till the set of right-angled numbers in order to consider (1) in such a set. Let us introduce the notion of right-angled numbers.

Definition. Right-angled number is such a non-negative real number, the square of which is a whole non-negative number.

The set of right-angled numbers $\mathbf{P}=\{0,1, \sqrt{ } 2, \sqrt{ } 3,2, \sqrt{ } 5, \ldots\}$ is countable. The system of right-angled numbers $\mathbf{P}=\langle\mathbf{P},+, \cdot, 0,1\rangle$ is defined by operations of addition and multiplication with two singled out elements (zero and unit). The system $P$ is
non-closed in relation to addition. Notice that the set of non-negative whole numbers is a subset of the set of right-angled numbers. Then consider (1) on the 2-dimensional lattice of right-angled numbers $z$ with coordinates $x_{0}, y_{0}$ and norm $z^{2}=x_{0}^{2}+y_{0}^{2}$. This norm of right-angled numbers with two summands in it always consists of whole numbers. The minimal (non-zero) norm (standard) of right-angled numbers equals 1 .

To construct Fermat's equality (1) in right-angled numbers, let us produce two chains of continued proportions connected with each other by the norm $z^{2}=x_{0}{ }^{2}+y_{0}{ }^{2}$ :

$$
\begin{align*}
& z / x_{0}=x_{0} / k=k / k_{1}=\ldots=k_{n-3} / k_{n-2} \\
& z / y_{0}=y_{0} / l=l / l_{1}=\ldots=l_{n-3} / l_{n-2} \tag{2}
\end{align*}
$$

where natural indices of the last terms of each chain in (2) are getting from $n>2$. Continued proportions (2) yield the following formulae:

$$
\begin{align*}
& k z=x_{0}^{2}, k_{1} z=x_{0} k, k_{2} z=x_{0} k_{1}, \ldots, k_{n-2} z=x_{0} k_{n-3} \\
& l z=y_{0}^{2}, l_{1} z=y_{0} l, l_{2} z=y_{0} l_{1}, \ldots, l_{n-2} z=y_{0} l_{n-3} \tag{3}
\end{align*}
$$

$$
\begin{align*}
& x_{0}^{2}=k z=\left(k_{1} z / x_{0}\right) z, \quad x_{0}^{3}=k_{1} z^{2}=\left(k_{2} z / x_{0}\right) z^{2}, \ldots, x_{0}{ }^{n}=k_{n-2} z^{n-1} \\
& y_{0}^{2}=l z=\left(l_{1} z / y_{0}\right) z, \quad y_{0}^{3}=l_{1} z^{2}=\left(l_{2} z / y_{0}\right) z^{2}, \ldots, y_{0}^{n}=l_{n-2} z^{n-1} \tag{4}
\end{align*}
$$

Now it is necessary to fix the norm for the partition of $z^{n}$ into two like powers in (1). As in the case of Beal's Conjecture, let us assume that $z, x, y$ in presupposed equality (1) have a common factor $d$, i. e., $z=\left(z^{\prime} d\right), x=\left(x^{\prime} d\right), y=\left(y^{\prime} d\right)$, where $z^{\prime}, x^{\prime}$, $y^{\prime}$ coprime. Thereupon we divide equality (1) by $z^{n-1}$ and get:
$z=\left(z^{\prime} d\right)=\left(x^{\prime} d\right)^{n} /\left(z^{\prime} d\right)^{n-l}+\left(y^{\prime} d\right)^{n} /\left(z^{\prime} d\right)^{n-l}=k+l$, where $k$ and $l$ integers if $d=\left(z^{\prime}\right)^{n-l}$ as a minimum, $d$ may be any whole number divisible by its minimum. From this and (3)-(4) it follows that $z^{2}=x_{0}{ }^{2}+y_{0}{ }^{2}$ and $z^{n}=z^{n-2}\left(x_{0}{ }^{2}+y_{0}{ }^{2}\right)$ as a scaled-up modification of the norm $z^{2}=x_{0}{ }^{2}+y_{0}^{2}$.

Further, one can get a singular partition of $z^{n}$ into three terms from (4) for the given norm when $n>2$ :

$$
\begin{equation*}
z^{n}=x_{0}^{n}+y_{0}^{n}+\lambda_{n} \tag{5}
\end{equation*}
$$

where $\lambda_{n}=z^{n-1}\left[\left(k-k_{n-2}\right)+\left(l-l_{n-2}\right)\right]$ is a remainder after subtracting $x_{0}{ }^{n}$ and $y_{0}{ }^{n}$ out of $z^{n}$ such that $\lambda_{n}>0$ when $n>2$ and $x_{0} y_{0} \neq 0, \lambda_{n}=0$ when $n=2$ and $x_{0} y_{0} \neq 0$,
$x_{0}, y_{0} \in[0, z], z \in(0, \infty)$.

Partitions (5) can be reduced to the norm, from which they were obtained:

$$
\begin{equation*}
z^{n}=x_{0}^{n}+y_{0}^{n}+\lambda_{n}=z^{n-2}\left(x_{0}^{2}+y_{0}^{2}\right)=x^{n}+y^{n} \tag{6}
\end{equation*}
$$

Formula (6) represents by itself a combinatorial equality of two partitions in three and two terms. In fact, it is one and the same partition in two terms. If it would not be so, equality (1) could not have the same norm and chain of proportions, from which (5) was obtained, would be different from (2). It means that partitions (6) are equal similar partitions of $n$-dimensional cube into two and three smaller $n$-dimensional cubes (however in general case, parallelepipeds $z^{n-2} x_{0}{ }^{2}$ and $z^{n-2} y_{0}{ }^{2}$ are not similar to
$n$-dimensional cubes and cannot form equal partitions; besides, irrational $x_{0}, y_{0}, x$, and $y$ cannot equate terms in (6)). One-to-one correspondence between partitions (6) is established by (2) and forms a closing stage in reverse problem method applied in mathematical psychology when Fermat's Last Theorem is considered as one of reverse problems for solution of mathematical tasks [1;2;3].

Thus there is isomorphism of partitions (6), owing to which scaling invariance of the norm $z^{n-2}\left(x_{0}{ }^{2}+y_{0}^{2}\right)$ leads to the next equalities of different fragments of partitions (6):

$$
\begin{equation*}
x_{0}{ }^{n}+y_{0}{ }^{n}=\left(x^{n} \text { or } y^{n}\right) \tag{7}
\end{equation*}
$$

and correspondingly $\lambda_{n}=\left(y^{n}\right.$ or $\left.x^{n}\right)$. It can be noticed that $x_{0}{ }^{n} \neq z^{n-2} \cdot y_{0}{ }^{2}=y^{n}$ and $y_{0}{ }^{n} \neq z^{n-2} \cdot x_{0}^{2}=x^{n} \quad$ because of the lack of coincidence of decompositions in factorization of numbers $x_{0}{ }^{n}$ and $y^{n}, y_{0}{ }^{n}$ and $x^{n}$. Obviously, $x_{0}{ }^{n} \neq z^{n-2} \cdot x_{0}{ }^{2}$ and $y_{0}{ }^{n}$ $\neq z^{n-2} \cdot y_{0}^{2}$.

Let us show now that $x_{0}$ and $y_{0}$ cannot be irrational in (7) on account of integer partition of $z^{n}$ into $x^{n}$ and $y^{n}$. Here two cases can occur: when $n$ is an odd number (designate it by $v=n_{\text {odd }} \geq 3$ ) and when $n$ is an even number (designate it by $\mu=$ $\left.n_{\text {even }} \geq 4\right)$. Considering the first case we find that $x_{0}$ and $y_{0}$ cannot be irrational in (9) as irrational square roots do not give a rational number in sum.

Let us consider the second case when $n=\mu$. Indeed, from the one hand, there is Pythagorean triple of numbers $z^{m}, x^{m}, y^{m}$ with $m=\mu / 2$ such that $\left(z^{m}\right)^{2}=\left(x^{m}\right)^{2}+\left(y^{m}\right)^{2}$. On the other hand, the initial equality can be written in the form $z^{2}=x_{0}{ }^{2}+y_{0}{ }^{2}$ showing that the indicated triple of numbers corresponds to the triple $z, x_{0}, y_{0}$ describing the like right-angled triangle. Therefore $z^{m} / x^{m}=z / x_{0}, z^{m} / y^{m}=z / y_{0}$, $x^{m}=x_{0} \cdot z^{m-1}, \quad y^{m}=y_{0} \cdot z^{m-1} \quad$ and $x_{0}$ and $y_{0}$ are not irrational.

So, it was revealed, as a result of the previous calculation, that equality (7) consists of whole numbers. Furthermore, Fermat's triple obtained from them for the given $n>2$, for example, $x_{0}, y_{0}, x$, is not the same by relative value as Fermat's triple $\quad x, y, z$ from (1), since $x_{0} / y_{0} \neq x / y$ that is clear from the following: $x_{0}{ }^{2} / y_{0}{ }^{2}=x^{n} / y^{n}=\left(x^{2} / y^{2}\right)\left(x^{n-2} / y^{n-2}\right)$.

Hence equality (7) represented in the form (1) describes another right-angled triangle different from that defined by Pythagorean triple $x_{0}, y_{0}, z$.

Let us come back to the assumption at the beginning of the proof that integer solution (1) exists. This assumption is substantiated only when there is a concrete solution (7) in whole numbers. In order to check validity of (7) it is necessary to do the same discourse as before, since equations (1) and (7) are identical by their properties. This procedure can be continued to infinity in the direction of decreasing whole numbers under condition that sequence of different chained equalities never stops and numbers $x_{0}{ }^{2}$ and $y_{0}{ }^{2}$ in (6) will be always whole. If it is not so and numbers in chained equalities (7) turn out to be fractions, then this means that solution (1) does not exist among whole numbers. On the other hand, infinite sequence of chained equalities (7) leads to infinite decreasing of positive whole numbers that is impossible and therefore assuming that there exists an integer solution of (1) when $n>2$ is not true. Thus the theorem is proved both for all even and for all odd degrees of whole numbers.

## III. Solution of Beal's Conjecture as a reverse problem of mathematical psychology.

The Beal conjecture states [4]:
The equation $A^{x}+B^{y}=C^{z}$ has no solution in positive integers $A, B, C, x, y$, and $z$ with $x, y$, and $z$ at least 3 and $A, B$, and $C$ coprime.

Or, restated [4]:
Let $A, B, C, x, y$, and $z$ be positive integers with $x, y, z>2$. If $A^{x}+B^{y}=C^{a}$, then $A, B$, and $C$ have a common factor.

Let us rewrite the Beal conjecture equality in the following way:

$$
\begin{equation*}
x^{n}+y^{n}=z^{n} \tag{8}
\end{equation*}
$$

with positive integers $x, y, z$ having a common factor and exponent $n$ taking simultaneously the next spectrum of values: $n=(k, l, m)$, where integers $k, l, m$
at least 3 and $n$ has one independent value for each term. Then we assume at the beginning that equality ( 8 ) exists and can explore some arbitrary solutions of equation (8) in whole numbers.

Consider equality (8) as a partition of $z^{n}$ into two parts $x^{n}$ and $y^{n}$ written in whole numbers. It can be reduced to the form of Pythagorean equation in real numbers dividing (8) by $z^{n-2}$. But for the purpose of integer computing and getting similar partitions from (8) with whole parts in it, it is necessary to use specific numbers. To produce such scaling in arithmetic geometry, let us use right-angled numbers (see above Definition).

One can rewrite (8) as an equality for some coprime $x^{\prime}, y^{\prime}, z^{\prime}$ and common whole factor $d:\left(z^{\prime} d\right)^{m}=\left(x^{\prime} d\right)^{k}+\left(y^{\prime} d\right)^{l}$ and fulfil scaling-down:
$\left(z^{\prime} d\right)^{2}=\left(x^{\prime} d\right)^{k} /\left(z^{\prime} d\right)^{m-2}+\left(y^{\prime} d\right)^{l} /\left(z^{\prime} d\right)^{m-2}=\left(x^{\prime}\right)^{k} d^{k-m+2} /\left(z^{\prime}\right)^{m-2}+\left(y^{\prime}\right)^{l} d^{l-m+2} /\left(z^{\prime}\right)^{m-2}$ $=x_{0}{ }^{2}+y_{0}{ }^{2}$, where $x_{0}{ }^{2}$ and $y_{0}{ }^{2}$ are squares of some right-angled numbers $x_{0}$ and $y_{0}$ with appropriate $d$. To get whole parts in the sum of this equality, one must regard exponents $(k-m+2)$ and $(l-m+2)$ with base $d$ equal to $\left(z^{\prime}\right)^{m-2}$. Obviously, $k$ and $l$ have to be more or equal $m-1$. If $k$ or $l$ does not satisfy this rule, then equality (8) cannot be represented on the lattice of right-angled numbers and consequently constructed in natural numbers. However, if ( $k, l$ ) $\geq m-1$, equality (8) assumes the following character (quantic) after fulfilling scaling-up:

$$
\begin{equation*}
z^{m}=x^{k}+y^{l}=z^{m-2}\left(x_{0}^{2}+y_{0}^{2}\right) \tag{9}
\end{equation*}
$$

To construct binomial (9) in right-angled numbers, let us apply the ancient method of getting powers of whole numbers using Euclid's theorem about proportional means and produce two chains of proportions connected with each other with some equality presenting integer $z$ as a sum of two whole numbers:

$$
\begin{align*}
& z / x=x / k=k / k_{l}=\ldots=k_{m-3} / k_{m-2}  \tag{10}\\
& z / y=y / l=l / l_{1}=\ldots=l_{m-3} / l_{m-2}
\end{align*}
$$

where $z, x, y$ are some unknown integers from (8), $m$ natural index at least 3 , and $z=$ $k_{m-2}+l_{m-2}$, where $k_{m-2}$ and $l_{m-2}$ with natural indices are some whole parts of $z$ taken from the method of scaling-down (see lower).

From proportions (3) one can obtain the next formulae:
$x^{2}=k z=\left(k_{1} z / x\right) z, x^{3}=k_{1} z^{2}=\left(k_{2} z / x\right) z^{2}, \ldots, x^{m}=k_{m-2} z^{m-1}$, $y^{2}=l z=\left(l_{1} z / y\right) z, y^{3}=l_{1} z^{2}=\left(l_{2} z / y\right) z^{2}, \ldots, y^{m}=l_{m-2} z^{m-1}$,
and get $x^{m}=\left(z k_{m-2}\right) z^{m-2}, \quad y^{m}=\left(z l_{m-2}\right) z^{m-2}, \quad$ where $k_{m-2}$ and $l_{m-2}$ are found from the basic equality (8):

$$
z=\left(z^{\prime} d\right)=\left(x^{\prime} d\right)^{k} /\left(z^{\prime} d\right)^{m-1}+\left(y^{\prime} d\right)^{l} /\left(z^{\prime} d\right)^{m-1}=k_{m-2}+l_{m-2}
$$

Then exponents $k$ and $l$ have to be more or equal $m$, if $k_{m-2}$ and $l_{m-2}$ are to be whole with $d=\left(z^{\prime}\right)^{m-1}$ as a minimum.

Now count that $z k_{m-2}=x_{0}{ }^{2}, z l_{m-2}=y_{0}{ }^{2}$, where $x_{0}, y_{o}$ are right-angled numbers from (2) when $d=\left(z^{\prime}\right)^{m-1}$, and get $x^{m}=x_{0}{ }^{2} z^{m-2}, y^{m}=y_{0}{ }^{2} z^{m-2}$. Hence square roots of $x^{m}, y^{m}$ are proportional means between $x_{0}{ }^{2}$ and $z^{m-2}, y_{0}{ }^{2}$ and $z^{m-2}$. Furthermore, relations (11) give only one-valued powers in partition (9), i.e., $x^{m}=x^{k}, y^{m}=y^{l}$. Thus we equalized degrees $k$ and $l$ to $m$ in (9) and got the following identity for the equal similar partitions of $z^{n}$ into two whole parts:

$$
\begin{equation*}
z^{m}=x^{m}+y^{m}=z^{m-2}\left(x_{0}{ }^{2}+y_{0}^{2}\right)=x^{k}+y^{l} \tag{12}
\end{equation*}
$$

where $x^{k}=\left(x^{k / m}\right)^{m}=x^{m}, y^{l}=\left(y^{l / m}\right)^{m}=y^{m}$, i.e., $k, l$ cannot be more or less than $m$ in order to satisfy boundaries of the right-angled lattice. Therefore $(k, l)=m$, since roots with degrees $m \geq 3$ cannot be numbers of the right-angled lattice and bases $x$, $y$ may be only whole powers beginning with exponent 1 under $m$. In other words, $m$ serves as a special quantifier for degrees of equation (8). Here also was used a closing stage in reverse problem method applied to the Beal conjecture.

All this yields that (8) comes to the Fermat equality in integers:

$$
\begin{equation*}
x^{m}+y^{m}=z^{m}, \quad m \geq 3 \tag{13}
\end{equation*}
$$

where common whole factor $d$ for $x=x$ ' $d$ and $y=y^{\prime} d$ may be any integer, for example, prime number. Then (13) can be reduced to the hypothetical equality in coprimes, which is impossible according to the above proof of Fermat's Last Theorem (see Lemma).

## IV. Conclusion. The method of reverse problem as new way in proving mathematical truths.

Summarizing the obtained proof of both Beal's Conjecture and Fermat's Last Theorem, let us single out those essential moments that properly make a reverse problem in such tasks. In proving Fermat's Last Theorem there exists one-to-one correspondence between partitions into two and three summands, or, in other words, between hypothetical Fermat's equality and the built partition into three summands there is isomorphism (preservation of Fermat's equality structure). Reducing of obtained partitions to the form of Fermat's equation just comes to a reverse problem of mathematical simulation of sought-for equality for higher powers of whole numbers.

The reverse problem method has three standard stages of mathematical modeling. In application to construction of the Fermat equation these stages look as follows. The first stage is when hypothetical Fermat's equality presents itself in the form of scaled-up norm $z^{n-2}\left(x_{0}^{2}+y_{0}^{2}\right)$, from which partitions in three summands
are obtained for each $n>2$. The second stage is when isomorphism between Fermat's equality and the built partition into three summands is established, i.e., there is preservation of operations, order, and topology of $n$-dimensional arithmetic space, where $n$ natural number. The third stage is when using final view of partitions one can get the initial image of the presupposed Fermat equation.

The method of reverse problem is applied also to Beal's Conjecture as generalized Fermat's Last Theorem. At the first stage the norm for the Beal equality is established. At the second stage the scaled-up norm equalizes powers of the Beal equality and the Fermat equality. At the third stage the concrete values for $x$ and $y$ in equalities (12) are determined.

## References.

1. Ивлиев Ю. А. Новые математические методы в психологии, их разработка и применение (проблемное исследование) // Психологический журнал 1988 т. 9 №1, 103-113. ISSN 0205-9592.
2. Ivliev Y. A. Reconstruction of nativus proof of Fermat's Last Theorem, (in Russian, title and abstract in English) // The Integrated Scientific Journal 2006 № 7, 3-9. ISSN 1729-3707.
3. Ивлиев Ю. А. Великая теорема Ферма как обратная задача математической психологии, (in Russian) // Materials of XI International Research and Practice Conference "Modern Scientific Potential - 2015", vol. 34, 32-36. Science and Education Ltd. Sheffield UK 2015. ISBN 978-966-8736-05-6.
4. Mauldin R. D., A generalization of Fermat's Last Theorem: the Beal conjecture and prize problem, Notices of the $A M S 44$ (1997), 1436-1437.
5. Wiles A., Modular elliptic curves and Fermat's Last Theorem, Ann. Math. 141 (1995), 443-551.
6. URL: http://ams.org/profession/prizes-awards/ams-supported/beal-prize

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