Solution of Beal's Conjecture as Mathematical Discovery *

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Abstract

The Beal conjecture is proved by arithmetic geometry methods known yet to ancient mathematicians and expanded with help of the reverse problem method used in applied mathematical research. These methods include constructing powers of whole numbers by means of proportions, making up partitions from them, their scaling-up and scaling-down in order to get equal similar partitions. As a result of such transformations, the Beal equation comes to the Fermat equation, which has no solution in positive whole numbers that is proved by the same methods plus Fermat's method of infinite descent. The given research is fulfilled in the system of right-angled numbers introduced by the author. All of it constitutes the mathematical discovery of Beal's Conjecture solution.

Keywords: Beal's Conjecture solution, Fermat's Last Theorem, arithmetic geometry, additive number theory: partitions, ancient mathematics

1. Introduction. Beal's Conjecture as generalized Fermat's Last Theorem.

Beal's Conjecture [1] deals with arbitrary powers of whole numbers combined in one equation similarly to the well-known equation of Fermat's Last Theorem. The Beal proposition can be solved by the ancient Greek arithmetic geometry methods applied successfully as well to the Fermat problem [2]. Among all well-known mathematics conjectures Beal's Conjecture is occupying a peculiar place being a generalization of Fermat's Last Theorem [1]. However the generalization in [1] concerns only the formal record of this conjecture and does not summarize the methods of proving Fermat's Last Theorem. On the contrary, the Beal conjecture comes to the Fermat problem considered as an arithmetic geometry problem and has easy simple solution obtained by additive number theory methods apparently available to ancient mathematicians and Fermat too [2]. The given proof of Beal's Conjecture can be related to the part of number theory defined as arithmetic algebraic geometry.

2. New arithmetic geometry of Beal's Conjecture and Fermat's Last Theorem (solution of both).

The Beal conjecture states [1]:

The equation $A^x + B^y = C^z$ *has no solution in positive integers A, B, C, x, y, and z with x, y, and z at least 3 and A,B, and C coprime.*

Or, restated [1]:

Let A, B, C, x, y, and z be positive integers with x, y, z > 2. If $A^x + B^y = C^z$*, then A, B, and C have a common factor.*

Let us rewrite the Beal conjecture equality in the following way:

$$x^n + y^n = z^n \qquad (1)$$

with positive integers x, y, z having a common factor and exponent n taking simultaneously the next spectrum of values: $n = (k, l, m)$, where integers k, l, m

at least 3 and n has one independent value for each term. Thus we assume at the beginning that equality (1) exists. Then we can explore some arbitrary solutions of equation (1) in whole numbers.

2.1. Beginning of Beal's Conjecture solution.

Consider equality (1) as a partition of z^n into two parts x^n and y^n written in whole numbers. It resembles the Pythagorean equation in real numbers, if we could reduce (1) to the degree 2 with whole parts in it. To produce such scaling, let us introduce the notion of right-angled numbers.

Definition. Right-angled number is such a non-negative real number, the square of which is a whole non-negative number.

The set of right-angled numbers **P** = {0, 1, √2 , √3 , 2, √5, ...} is countable. The system of right-angled numbers P = ⟨**P**,+,·,0,1⟩ is defined by operations of addition and multiplication and two singled out elements (zero and unit). The system P is non-closed in relation to addition. Notice that the set of non-negative whole numbers is a subset of the set of right-angled numbers. Then consider (1) on the 2-dimensional lattice of right-angled numbers with coordinates x_o , y_o and norm $z^2 = x_o^2 + y_o^2$. The norm of right-angled numbers is always whole and cannot be less than 1.

For the purpose of reducing (1) to the view of Pythagorean equation in the system of right-angled numbers, one can rewrite (1) as an equality for some coprime x', y', z' and common whole factor d : $(x'd)^k + (y'd)^l = (z'd)^m$ and fulfill scaling-down:

$(z'd)^2 = (x'd)^k / (z'd)^{m-2} + (y'd)^l / (z'd)^{m-2} = (x')^k d^{k-m+2} / (z')^{m-2} + (y')^l d^{l-m+2} / (z')^{m-2} =$

$x_o^2 + y_o^2$, where x_o^2 and y_o^2 with appropriate d are squares of some right-angled numbers x_o and y_o. To get whole parts in the sum of this equality, one must regard exponents $(k-m+2)$ and $(l-m+2)$ with base d equal to $(z')^{m-2}$. Obviously, k and l have to be more or equal $m-1$. If k or l does not satisfy this rule, then equality (1) cannot be represented on the lattice of right-angled numbers and consequently constructed from natural numbers on this lattice. However, if $(k, l) \geq m-1$, equality (1) assumes the following character (quantic) after fulfilling scaling-up:

$$z^m = x^k + y^l = z^{m-2} (x_o^2 + y_o^2) \qquad (2)$$

Now let us apply the ancient method of getting powers of whole numbers [2] (using Euclid's geometrical theorem) and produce two chains of proportions connected with each other with some equality presenting integer z as a sum of two whole numbers:

$$z/x = x/k = k/k_1 = \dots = k_{m-3} /k_{m-2} \qquad (3)$$

$$z/y = y/l = l/l_1 = \ldots = l_{m-3}/l_{m-2}$$

where z, x, y are some unknown integers from (1) when their degrees are reduced to m, i.e., $x^k = (x^{k/m})^m$, $y^l = (y^{l/m})^m$ (for the reason of simplicity we do not change the designations for x, y), m natural index at least 3, and $z = k_{m-2} + l_{m-2}$, k_{m-2} and l_{m-2} some whole parts of z taken from the method of scaling-down (see below).

From proportions (3) one can obtain the next formulae:

$$x^2 = kz = (k_1z/x)z\,,\ x^3 = k_1z^2 = (k_2z/x)z^2\,,\ \ldots\,,\ x^m = k_{m-2}\,z^{m-1}, \tag{4}$$

$$y^2 = lz = (l_1z/y)z\,,\ y^3 = l_1z^2 = (l_2z/y)z^2\,,\ \ldots\,,\ y^m = l_{m-2}\,z^{m-1},$$

and get $x^m = (zk_{m-2})z^{m-2}$, $y^m = (zl_{m-2})z^{m-2}$, where k_{m-2} and l_{m-2} are found from the basic equality (1):

$$z = (z'd) = (x'd)^k/(z'd)^{m-1} + (y'd)^l/(z'd)^{m-1} = k_{m-2} + l_{m-2}$$

and exponents k and l have to be more or equal m, if k_{m-2} and l_{m-2} are to be whole with $d = (z')^{m-1}$ as a minimum (d can be any whole number divisible by this minimum).

Now count that $zk_{m-2} = x_o^2$, $zl_{m-2} = y_o^2$, where x_o, y_o are right-angled numbers from (2) when $d = (z')^{m-1}$, and get $x^m = x_o^2\,z^{m-2}$, $y^m = y_o^2\,z^{m-2}$. Hence square roots of x^m, y^m are proportional means between x_o^2 and z^{m-2}, y_o^2 and z^{m-2}. Furthermore, relations (4) give only one-valued powers in partition (1), i.e., $x^m = x^k$, $y^m = y^l$. Thus we equalized degrees k and l to m in the quantic (2) and got the following identity for the equal similar partitions of z^n into two whole parts:

$$z^m = x^m + y^m = z^{m-2}(x_o^2 + y_o^2) = x^k + y^l \tag{5}$$

where $x^k = (x^{k/m})^m = x^m$, $y^l = (y^{l/m})^m = y^m$, i.e., k, l cannot be more or less than m in order to satisfy boundaries of the right-angled lattice. Therefore $(k, l) = m$, since roots with degrees $m \geq 3$ cannot be numbers of the right-angled lattice, and bases x, y may be only whole powers beginning with exponent 1 under m. In other words, m serves as a special quantifier for degrees of equation (1).

This yields that (1) comes to the Fermat equality in integers:

$$x^m + y^m = z^m, \; m \geq 3 \qquad (6)$$

where $x = x'd$, $y = y'd$, $z = z'd$, d may be any whole factor, in particular prime factor. Then (6) can be reduced to the hypothetical equality in coprimes, which is impossible according to Fermat's Last Theorem. Now one can prove Fermat's Last Theorem with the same methods in order to fulfill solution of the Beal conjecture in full and one measure.

2.2. Completion of Beal's Conjecture solution.

Proof of Fermat's Last Theorem. Fermat's Last Theorem claims that the following equation (7) with integers z, x, y and natural exponent $n > 2$ has no solution:

$$z^n = x^n + y^n \qquad (7)$$

Let us check this assertion. Suppose however that at least one solution was found. Then we shall try to construct such a solution and make certain of its impossibility. We shall work in the system of right-angled numbers (see above *Definition*).

Consider (7) on the 2-dimensional lattice of right-angled numbers with right-angled coordinates x_0 , y_0 and norm $z^2 = x_0^2 + y_0^2$ differing by its square fragments relating to definite right-angled coordinates and being a partition of number z^2 into two summands represented by non-negative whole numbers. The minimal (non-zero) norm (standard) of right-angled numbers equals 1.

To construct powers of whole numbers presented in (7), let us produce two chains of continued proportions connected with each other by the norm $z^2 = x_0^2 + y_0^2$:

$$z/x_0 = x_0/k = k/k_1 = \ldots = k_{n-3}/k_{n-2}$$

$$z/y_0 = y_0/l = l/l_1 = \ldots = l_{n-3}/l_{n-2} \qquad (8)$$

where natural indices of the last terms of each chain in (8) are getting from $n > 2$. Continued proportions (8) yield the following formulae:

$$kz = x_0^2,\ k_1z = x_0k,\ k_2z = x_0k_1,\ \ldots,\ k_{n-2}z = x_0k_{n-3}$$

$$lz = y_0^2,\ l_1z = y_0l,\ l_2z = y_0l_1,\ \ldots,\ l_{n-2}z = y_0l_{n-3} \qquad (9)$$

$$x_0^2 = kz = (k_1z/x_0)z,\quad x_0^3 = k_1z^2 = (k_2z/x_0)z^2,\ \ldots,\ x_0^n = k_{n-2}z^{n-1}$$

$$y_0^2 = lz = (l_1z/y_0)z,\quad y_0^3 = l_1z^2 = (l_2z/y_0)z^2,\ \ldots,\ y_0^n = l_{n-2}z^{n-1} \qquad (10)$$

Now it is necessary to fix the norm for the partition of z^n into two like powers in (7). As in the case of Beal's Conjecture, let us assume that z, x, y in presupposed equality (7) have a common factor d, i. e., $z = (z'd)$, $x = (x'd)$, $y = (y'd)$, where z', x'. y' coprime. Thereupon we divide equality (7) by z^{n-1} and get:

$z = (z'd) = (x'd)^n/(z'd)^{n-1} + (y'd)^n/(z'd)^{n-1} = k + l$, where k and l integers if $d = (z')^{n-1}$ as a minimum (d can be any whole number divisible by this minimum). From this and (9)-(10) it follows that $z^2 = x_0^2 + y_0^2$ and $z^n = z^{n-2}(x_0^2 + y_0^2)$ is a scaled-up modification of the norm $z^2 = x_0^2 + y_0^2$.

Further, one can get a singular partition of z^n into three terms from (10) for the given norm when $n > 2$:

$$z^n = x_0^{\,n} + y_0^{\,n} + \lambda_n \qquad (11)$$

where $\lambda_n = z^{n-1}\,[\,(k - k_{n-2}) + (l - l_{n-2})\,]$ is a remainder after subtracting $x_0^{\,n}$ and $y_0^{\,n}$ out of z^n such that $\lambda_n > 0$ when $n > 2$ and $x_0\, y_0 \neq 0$, $\lambda_n = 0$ when $n = 2$ and $x_0\, y_0 \neq 0$,

$x_0\,,\ y_0, \in [0, z]$, $z \in (0, \infty)$.

Lemma. There exists one-to-one correspondence between each pair of whole numbers ($x_0\,,\ y_0$) with norm $z^2 = x_0^{\,2} + y_0^{\,2}$ from 2-dimensional arithmetic space and each corresponding partition of any whole degree $n > 2$ of integer z from n-dimensional arithmetic space into the sum of the same degrees of whole numbers $x_0\,,\ y_0$ and remainder λ_n from (11).

Proof. Isomorphism between the set of points of 2-dimensional Euclidean space with position vector length z and coordinates $x_0\,,\ y_0$, the set of partitions of z^2 into squares, and the sets of partitions (11) for any whole $n > 2$ can be written as follows : $\{z \Rightarrow (xo\,, yo)\} \leftrightarrow \{\ z^2 = x_0^{\,2} + y_0^{\,2}\ \} \leftrightarrow \{\ z^n = x_0^{\,n} + y_0^{\,n} + \lambda_n\}$,

where sets are generated by the next degree similarities:

$z \leftrightarrow z^2 \leftrightarrow z^n$, $x_0 \leftrightarrow\ x_0^{\,2} \leftrightarrow x_0^{\,n}$, $y_0 \leftrightarrow y_0^{\,2} \leftrightarrow y_0^{\,n}$, and is proved by proportions (8). It should be noticed that irrational right-angled x_0 , y_0 cannot equate terms in one-to-one correspondence formulae and do not form the indicated isomorphism for two last partitions.

Partitions (11) can be reduced to the norm, from which they were obtained:

$$z^n = x_0^{\,n} + y_0^{\,n} + \lambda_n = z^{n-2}\,(\,x_0^{\,2} + y_0^{\,2}\,) = x^n + y^n \qquad (12)$$

Formula (12) represents by itself a combinatorial equality of two partitions in three and two terms. If it would not be so, equality (7) could not have the same norm and chains of proportions, from which (12) was obtained, would be different from (8). In the case of right-angled numbers this equality is realized only when x_0, y_0 integers.

Thus scaling invariance of the norm $z^{n-2}\,(x_0^2 + y_0^2)$ leads to the next equalities of different fragments of partitions (12):

$$x_0^n + y_0^n = (x^n \text{ or } y^n) \qquad (13)$$

and correspondingly $\lambda_n = (y^n \text{ or } x^n)$. It can be noticed that $x_0^n \neq z^{n-2}\cdot y_0^2 = y^n$ and $y_0^n \neq z^{n-2}\cdot x_0^2 = x^n$ because of the lack of coincidence of decompositions in factorization of numbers x_0^n and y^n, y_0^n and x^n. Obviously, $x_0^n \neq z^{n-2}\cdot x_0^2$ and $y_0^n \neq z^{n-2}\cdot y_0^2$.

Let us show now that x_0 and y_0 cannot be irrational in (13) on account of integer partition of z^n into x^n and y^n. Here two cases can occur: when n is an odd number (designate it by $\nu = n_{\text{odd}} \geq 3$) and when n is an even number (designate it by $\mu = n_{\text{even}} \geq 4$). Considering the first case we find that x_0 and y_0 cannot be irrational in (13) as irrational square roots do not give a rational number in sum.

Let us consider the second case when $n = \mu$. Indeed, from the one hand, there is Pythagorean triple of numbers z^m, x^m, y^m with $m = \mu/2$ such that $(z^m)^2 = (x^m)^2 + (y^m)^2$. On the other hand, the initial equality can be written in the form $z^2 = x_0^2 + y_0^2$ showing that the indicated triple of numbers corresponds to the triple z, x_0, y_0 describing the like right-angled triangle. Therefore $z^m/x^m = z/x_0$, $z^m/y^m = z/y_0$, $x^m = x_0\cdot z^{m-1}$, $y^m = y_0\cdot z^{m-1}$ and x_0 and y_0 are not irrational.

So, it was revealed, as a result of the previous calculation, that equality (13) consists of whole numbers. Furthermore, Fermat's triple obtained from them for the given $n > 2$, for example, x_0, y_0, x, is not the same by value as Fermat's triple x, y, z from (7), since $x_0 / y_0 \neq x / y$ that is clear from the following: $x_0^2/y_0^2 = x^n/y^n = (x^2/y^2)(x^{n-2}/y^{n-2})$.

Hence equality (13) represented in the form (12) describes another right-angled triangle different from that defined by Pythagorean triple x_0 , y_0 , z .

Let us come back to the assumption at the beginning of the proof that integer solution (7) exists. This assumption is substantiated only when there is a concrete solution (13) in whole numbers. In order to check validity of (13) it is necessary to do the same discourse as before, since equations (7) and (13) are identical by their properties. This procedure can be continued to infinity in the direction of decreasing whole numbers under condition that sequence of different chained equalities never stops and numbers x_0^2 and y_0^2 in (12) will be always whole. If it is not so, i.e., x_0^2 and y_0^2 in chained equalities (13) turn out to be fractions, then this means that solution (7) does not exist in the system of right-angled numbers**.** Actually, since all partitions of the type (12) are built from the very beginning exclusively on the set of right-angled numbers' squares being in fact whole items of finite series of partitions, then non-whole x_0^2 and y_0^2 show pointlessness of such procedure, i.e., the absence of integer solution (7) or zero solution. On the other hand, infinite sequence of chained equalities (13) leads to infinite decreasing of positive whole numbers that is impossible and therefore assuming that there exists an integer solution of (7) when $n > 2$ is not true. Thus the theorem is proved both for all even and for all odd degrees of whole numbers.

2.3. Results and discussion.

So, the full proof of Beal's Conjecture is obtained owing to Fermat's method of infinite descent (see above). Materials and methods of this research are described in detail in the previous sections of the article. One should draw special attention to constructing chains of proportions (8), which lead to the basic equality of partitions (12). This equality is founded on the one-to-one correspondence described by *Lemma* and establishes isomorphism between partitions in two terms and partitions in three terms for each chain of proportions, i.e., for each pair of whole numbers x_0 , y_0 . It should be noticed that these partitions have specific geometrical view being partitions of n-dimensional cubes into smaller n-dimensional cubes, so they are not linear sections of segments representing whole powers z^n, x_0^n, y_0^n, x^n, y^n on the 2-dimensional lattice of right-angled numbers. Thus partitions in (12) are equal similar partitions of n-dimensional cubes when one of the parts of the two-termed partition is divided in two parts with help of x_0 , y_0 in n-dimensional arithmetical space.

3. Conclusion.

Returning to the full proof of Beal's conjecture, let us note the important circumstance, allowing to complete it, consisted in the true proof of Fermat's Last Theorem as particular case of the considered conjecture. So, Beal's Conjecture led to solving the centuries-old problem of mankind and can be called rightfully Generalized Fermat's Last Theorem in memory of the greatest discovery in the history of human science opening new ways in world cognition and understanding ancient knowledge by means of modern mathematical language using the notion of isomorphism of mathematical objects and the reverse problem method of applied mathematics [3;4;5]. Particularly, solution of Beal's Conjecture contains in itself the description of a new hypothetical mathematical object with simple properties conditioned only by its intrinsic structure. This object (represented by hypothetical Fermat's equality in whole numbers) has been unknown in pure mathematics till now and can be understood on the following scheme of reverse problem solving:

$$\begin{array}{ccc} z^n = x^n + y^n & \leftarrow & z^n = (x_0^n + y_0^n) + \lambda_n \\ \downarrow & & \uparrow \\ z^n = z^{n-2}(x_0^2 + y_0^2) & \rightarrow & z^n = x_0^n + y_0^n + \lambda_n \end{array}$$

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* Статья была направлена в журнал Arnold Mathematical Journal 09.11.2015 и определена как математически некорректная. Автор согласен с тем, что его терминология выходит за рамки общепринятой математической терминологии, но считает, что в проблемной области математических открытий допустимы авторские определения, не противоречащие логике математического исследования и смыслу предлагаемых операций. Например, экзотический термин right-angled numbers был введен для того, чтобы отличить эти числа от чисел прямого угла (right angled numbers), являющихся целыми числами прямого угла, рассматриваемого в области фигурных чисел как геометрическая фигура.