Ivliev Y. A.

HOW FERMAT COULD PROVE HIS OWN THEOREM AND BEAL'S CONJECTURE TOO. *

Doctor of Science, academician

International Informatization Academy, Moscow, Russian Federation

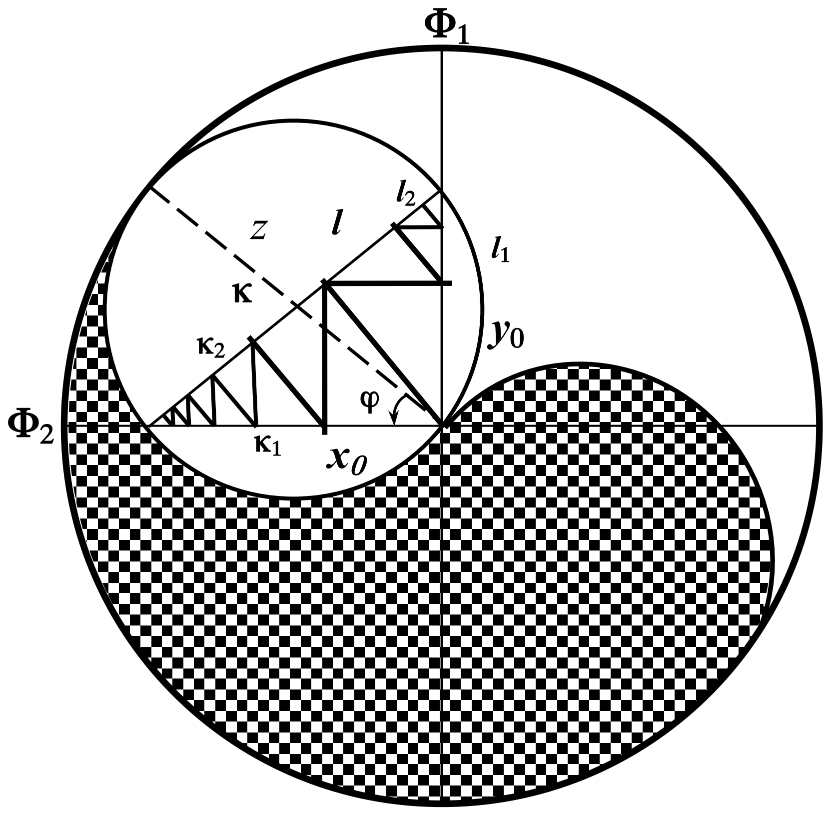
In this paper the Beal conjecture is solved by arithmetic geometry methods known yet to ancient mathematicians and expanded with help of the reverse problem method used in applied mathematical research. These methods could be implicitly used by Fermat in his creative search and led him to the discovery of a proper mathematical structure of his "Last Theorem". As a matter of fact, this discovery just determines exclusiveness of Fermat's Last Theorem and Beal's Conjecture proofs. In the result of this approach the Beal equation comes to the Fermat equation, which has no solution in positive whole numbers that is proved owing to Fermat's breakthrough in arithmetic geometry and Fermat's method of infinite descent.

Keywords: Beal's Conjecture solution, Fermat's Last Theorem, ancient mathematics, arithmetic geometry, additive number theory: partitions

1. Introduction. Beal's Conjecture as generalized Fermat's Last Theorem.

Beal's Conjecture [1] deals with arbitrary powers of whole numbers combined in one equation similarly to the well-known equation of Fermat's Last Theorem. The Beal proposition can be solved by the ancient Greek arithmetic geometry methods applied successfully as well to the Fermat problem [2]. Among all well-known mathematics conjectures Beal's Conjecture is occupying a peculiar place being a generalization of Fermat's Last Theorem [1]. However the generalization in [1] concerns only the formal record of this conjecture and does not summarize the methods of proving Fermat's Last Theorem. On the contrary, the Beal conjecture comes to the Fermat problem considered as an arithmetic geometry problem and has easy simple solution obtained by additive number theory methods apparently available to ancient mathematicians and Fermat too [2]. Solution of Beal's Conjecture is based on the discovery of a proper mathematical structure of Fermat's equation in his "Last Theorem" [2-5]. This structure is defined as Cartesian product of the set of real numbers in multidimensional arithmetical space and well described by Euclid's geometrical theorem (geometrical view of this theorem can be seen in Fig. 1 taken from [3]). Fermat, of course, knew it and could operate right angle triangles from Euclid's theorem in such a way that to represent his equation as a similar partition of Pythagorean equation in whole numbers. Now let us proceed considering Beal's Conjecture solution and Fermat's Last Theorem from the viewpoint of Euclid's geometrical theorem about mean proportionals.

Fig. 1(see designations in the text)



2. Arithmetic geometry of Beal's Conjecture and Fermat's Last Theorem (solution of both).

The Beal conjecture states [1]:

The equation $A^x + B^y = C^z$ has no solution in positive integers A, B, C, x, y, and z with x, y, and z at least 3 and A,B, and C coprime.

Or, restated [1]:

Let A, B, C, x, y, and z be positive integers with x, y, z > 2. If $A^x + B^y = C^z$, then A, B, and C have a common factor.

Let us rewrite the Beal conjecture equality in the following way:

$$x^n + y^n = z^n \qquad (1)$$

with positive integers *x, y, z* having a common factor and exponent *n* taking simultaneously the next spectrum of values: $n = (k, l, m)$, where integers *k, l, m*

at least 3 and *n* has one independent value for each term. Thus we assume at the beginning that equality (1) in whole numbers exists, or partitions of the type (1) can be obtained. This method of proof is related to plausible reasoning and called the rule of contraries. Then one can explore some arbitrary solutions of equation (1) in whole numbers.

2.1. Beginning of Beal's Conjecture solution.

Consider equality (1) as a partition of whole number z^n into two whole parts x^n and y^n with whole *x* and *y*. It resembles the Pythagorean equation in real numbers, if we could reduce powers in (1) to the degree 2 with whole parts in the same partition in order to check its value by unit steps and construct (1) from whole numbers. To produce such scaling, let us introduce the notion of right-angled numbers (at all appearance Fermat implicitly worked with it using Pythagorean triples of whole numbers as right-angled numbers in arithmetic geometry of ancients).

Definition. Right-angled number is such a non-negative real number, the square of which is a whole non-negative number.

The set of right-angled numbers **P** = {0, 1, √2 , √3 , 2, √5, …} is countable. The system of right-angled numbers P = ⟨**P**,+,·,0,1⟩ is defined by operations of addition and multiplication and two singled out elements (zero and unit). The system P is non-closed in relation to addition. Notice that the set of non-negative whole numbers is a subset of the set of right-angled numbers. Then consider (1) on the 2-dimensional lattice of right-angled numbers with coordinates x_o , y_o and that, which we call the norms of right-angled number z differing from each other by the value of its summands: $z^2 = x_o^2 + y_o^2$. The norm of right-angled numbers is always whole and cannot be less than 1. Whole numbers x_o^2 and y_o^2 run through values from 0 to z^2 and from z^2 to 0 one by one. So, number z has z^2 different partitions as its norms (among them Pythagorean triples can occur and Fermat applied Pythagorean equations as peculiar norms for number z in 2-dimensional Euclidean space).

For the purpose of reducing (1) to the view of Pythagorean equation in the system of right-angled numbers, one can rewrite (1) as an equality for some coprime x', y', z', and common whole factor d : $(x'd)^k + (y'd)^l = (z'd)^m$, and fulfill scaling-down:

$$(z'd)^2 = (x'd)^k / (z'd)^{m-2} + (y'd)^l / (z'd)^{m-2} = (x')^k d^{k-m+2} / (z')^{m-2} + (y')^l d^{l-m+2} / (z')^{m-2} =$$

$x_o^2 + y_o^2$, where x_o^2 and y_o^2 with appropriate d are squares of some right-angled numbers x_o and y_o. To get whole parts in the sum of this equality, one must regard exponents $(k-m+2)$ and $(l-m+2)$ for base d tuple to $(z')^{m-2}$. Obviously, k and l have to be more or equal $m-1$. If k or l does not satisfy this rule, then equality (1) cannot be represented on the lattice of right-angled numbers and consequently constructed from natural numbers on this lattice. However, if $(k, l) \geq m-1$, equality (1) assumes the following character (quantic) after fulfilling scaling-up:

$$z^m = x^k + y^l = z^{m-2} (x_o^2 + y_o^2) \qquad (2)$$

Now let us apply the ancient method of constructing powers using Euclid's geometrical theorem [2] and produce two chains of proportions connected with each other with some equality presenting integer z as a sum of two whole numbers:

$$z/x_o = x_o/k = k/k_1 = \ldots = k_{m-3}/k_{m-2} \qquad (3)$$

$$z/y_o = y_o/l = l/l_1 = \ldots = l_{m-3}/l_{m-2}$$

where z, x_o, y_o are right-angled numbers from (2), m natural index at least 3,and $z=k+l$; k and l are some whole parts of z taken from the method of scaling-down (see below).

From proportions (3) one can obtain the next formulae:

$$x_o^2 = kz = (k_1z/x_o)z\ ,\ x_o^3 = k_1z^2 = (k_2z/x_o)z^2\ , \ldots ,\ x_o^m = k_{m-2}\, z^{m-1}, \qquad (4)$$

$$y_o^2 = lz = (l_1z/y_o)z\ ,\ y_o^3 = l_1z^2 = (l_2z/y_o)z^2\ , \ldots ,\ y_o^m = l_{m-2}\, z^{m-1},$$

where k and l are found from the basic equality (1):

$$z = (z'd) = (x'd)^k/(z'd)^{m-1} + (y'd)^l/(z'd)^{m-1} = k + l$$

and exponents k and l have to be more or equal m, if numbers k and l would be whole with $d = (z')^{m-1}$ as a minimum (d can be any whole number divisible by this minimum).

For right-angled numbers x_o, y_o with d tuple to $(z')^{m-1}$, one can get from (2) and (4) the following powers: $x^m = x_o^2\, z^{m-2}$, $y^m = y_o^2\, z^{m-2}$, where square roots of x^m, y^m are proportional means between x_o^2 and z^{m-2}, y_o^2 and z^{m-2}, describing a bigger right angle triangle similar to that with sides x_o, y_o, z represented by x_o^2, y_o^2, z^2 of Pythagorean equation (see Fig. 1 from [3] with the same designations). If the sides of this bigger right angle triangle (and therefore the edges of n-dimensional cubes) express themselves in whole numbers, then x_o, y_o are whole too according to Euclid's theorem and right angle triangles' similarity. Furthermore, relations (2) - (4) give only one-valued powers in partition (1), i.e., $x^m = x^k$, $y^m = y^l$. Thus we equalized degrees k and l to m in the quantic (2) and got the following identity for the equal similar partitions of z^n into two whole parts:

$$z^m = x^k + y^l = z^{m-2}(x_o^2 + y_o^2) = x^m + y^m \qquad (5)$$

where $x^k = (x^{k/m})^m = x^m$, $y^l = (y^{l/m})^m = y^m$, i.e., k, l cannot be more or less than m in order to satisfy boundaries of the right-angled lattice. Therefore $(k, l) = m$, since roots with degrees $m \geq 3$ cannot be numbers of the right-angled lattice, and bases x, y may be only whole powers beginning with exponent 1 under m. In other words, m serves as a special quantifier for powers of equation (1). It means that the right angle triangle with whole sides $\surd\, x^m$, $\surd\, y^m$ can be obtained only from whole numbers x_o , y_o representing the similar right angle triangle.

Thus partition (1), similar to partitions $z = k + l$ and $z^m = z^{m-2}\ (x_o^2 + y_o^2)$, is possible only if integers x_o , y_o , z constitute a Pythagorean triple. So, in such a case right-angled numbers x_o , y_o may be called right angle numbers for the right angle considered as a geometrical figure. The entire transformation process of identical partition (1) can be viewed on the following diagram (Fig. 2):

Fig. 2 (see designations in the text)

$$\begin{array}{ccc} z^m = x^k + y^l & \rightarrow & z^{m-2}\ (x_o^2 + y_o^2) \\ \uparrow & & \downarrow \\ z^m = x^m + y^m & \leftarrow & (\,x_o^2\, z^{m-2}\,) + (\,y_o^2\, z^{m-2}\,) \end{array}$$

This yields that (1) comes to the Fermat equality in integers:

$$x^m + y^m = z^m,\ \ m \geq 3 \qquad (6)$$

where $x = x'd$, $y = y'd$, $z = z'd$, d may be any whole factor, in particular prime factor. Then (6) can be reduced to the hypothetical equality in coprimes, which is impossible according to Fermat's Last Theorem. Now one can prove Fermat's Last

Theorem with the same methods as above in order to fulfill solution of the Beal conjecture in full and one measure.

2.2. Completion of Beal's Conjecture solution.

Proof of Fermat's Last Theorem. Fermat's Last Theorem claims that the following equation (7) with integers z, x, y, and natural exponent $n > 2$ has no solution:

$$z^n = x^n + y^n \tag{7}$$

Let us check this assertion. Suppose however that at least one solution was found. Then we shall try to construct such a solution and make certain of its impossibility. We shall work in the system of right-angled numbers (see above Definition).

Consider (7) on the 2-dimensional lattice of right-angled numbers with right-angled coordinates x_0 , y_0 and corresponding norm $z^2 = x_0^2 + y_0^2$ differing by its square fragments relating to definite right-angled coordinates and being a partition of number z^2 into two summands represented by non-negative whole numbers. The minimal (non-zero) norm (standard) of right-angled numbers equals 1.

To construct powers of whole numbers presented in (7), let us produce two chains of continued proportions connected with each other by the norm $z^2 = x_0^2 + y_0^2$:

$$z/x_0 = x_0/k = k/k_1 = \ldots = k_{n-3}/k_{n-2}$$

$$z/y_0 = y_0/l = l/l_1 = \ldots = l_{n-3}/l_{n-2} \tag{8}$$

where natural indices of the last terms of each chain in (8) come of $n > 2$. Continued proportions (8) yield the following formulae:

$$kz = x_0^2,\ k_1z = x_0k,\ k_2z = x_0k_1,\ \ldots,\ k_{n-2}z = x_0k_{n-3}$$

$$lz = y_0^2,\ l_1z = y_0l,\ l_2z = y_0l_1,\ \ldots,\ l_{n-2}z = y_0l_{n-3} \tag{9}$$

$$x_0^2 = kz = (k_1z/x_0)z,\quad x_0^3 = k_1z^2 = (k_2z/x_0)z^2,\ \ldots,\ x_0^n = k_{n-2}z^{n-1}$$

$$y_0^{\,2} = lz = (l_1 z / y_0) z, \quad y_0^{\,3} = l_1 z^2 = (l_2 z / y_0) z^2, \ \ldots\ , \ y_0^{\,n} = l_{n-2} z^{n-1} \qquad (10)$$

Now it is necessary to fix the norm for the partition of z^n into two like powers in (7). As in the case of Beal's Conjecture, let us assume that z, x, y in presupposed equality (7) have a common factor d, i. e., $z = (z'd)$, $x = (x'd)$, $y = (y'd)$, where z', x'. y' coprime. Thereupon we divide equality (7) by z^{n-1} and get:

$z = (z'd) = (x'd)^n/(z'd)^{n-1} + (y'd)^n/(z'd)^{n-1} = k + l$, where k and l integers if $d = (z')^{n-1}$ as a minimum (d can be any whole number divisible by this minimum). From this and (9)-(10) it follows that $z^2 = x_0^{\,2} + y_0^{\,2}$ and $z^n = z^{n-2}\,(x_0^{\,2} + y_0^{\,2})$ is a scaled-up modification of the norm $z^2 = x_0^{\,2} + y_0^{\,2}$, where x_0, y_0 are some right-angled numbers.

Further, one can get a singular partition of z^n into three terms from (10) for the given norm when $n > 2$:

$$z^n = x_0^{\,n} + y_0^{\,n} + \lambda_n \qquad (11)$$

where $\lambda_n = z^{n-1}\,[\,(k - k_{n-2}) + (l - l_{n-2})\,]$ is a remainder after subtracting $x_0^{\,n}$ and $y_0^{\,n}$ out of z^n such that $\lambda_n > 0$ when $n > 2$ and $x_0\, y_0 \neq 0$, $\lambda_n = 0$ when $n = 2$ and $x_0\, y_0 \neq 0$,

$x_0, y_0 \in [0, z]$, $z \in (0, \infty)$.

Lemma. There exists one-to-one correspondence between each pair of right-angled numbers (x_0, y_0) with norm $z^2 = x_0^{\,2} + y_0^{\,2}$ from 2-dimensional arithmetic space and each corresponding partition of any whole power $n > 2$ of integer z from n-dimensional arithmetic space into the sum of the same powers of right-angled numbers x_0, y_0 and remainder λ_n from (11).

Proof. Isomorphism between the set of points of 2-dimensional Euclidean space with position vector length z and coordinates x_0, y_0, the set of partitions of z^2 into squares, and the sets of partitions (11) for any whole $n > 2$ can be written as follows :

$$\{z \Rightarrow (xo\,, yo)\} \leftrightarrow \{\, z^2 = x_0^2 + y_0^2 \,\} \leftrightarrow \{\, z^n = x_0^n + y_0^n + \lambda_n \},$$

where sets are generated by the next power similarities:

$z \leftrightarrow z^2 \leftrightarrow z^n$, $x_0 \leftrightarrow x_0^2 \leftrightarrow x_0^n$, $y_0 \leftrightarrow y_0^2 \leftrightarrow y_0^n$, and is proved by (8) – (11).

In the case of partition (7) partition (11) can be reduced to the norm, from which they were obtained:

$$z^n = x_0^n + y_0^n + \lambda_n = z^{n-2}\,(x_0^2 + y_0^2) = x^n + y^n \tag{12}$$

as there is one-to-one correspondence between partitions in (12).

Formula (12) represents by itself a combinatorial equality of two similar partitions in three and two terms. In the case of (7) this equality is realized only when x_0, y_0 integers. Indeed, scaling invariance of the norm $z^{n-2}\,(x_0^2 + y_0^2)$ leads to the next equalities of different fragments of partitions (12):

$$x_0^n + y_0^n = (\, x^n \; or \; y^n \,) \tag{13}$$

and correspondingly $\lambda_n = (y^n \; or \; x^n)$. It can be noticed that $x_0^n \neq z^{n-2}\cdot y_0^2 = y^n$ and $y_0^n \neq z^{n-2}\cdot x_0^2 = x^n$ because of the lack of coincidence of decompositions in factorization of numbers x_0^n and y^n, y_0^n and x^n. Obviously, $x_0^n \neq z^{n-2}\cdot x_0^2$ and $y_0^n \neq z^{n-2}\cdot y_0^2$. So, only one partition $z^n = (x_0^n + y_0^n) + \lambda_n$ as two terms one corresponds to

$z^n = x^n + y^n$.

Let us show now that x_0 and y_0 cannot be irrational in (13) on account of integer partition of z^n into x^n and y^n. Here two cases can occur: when n is an odd number (designate it by $\nu = n_{\text{odd}} \geq 3$) and when n is an even number (designate it by $\mu = n_{\text{even}} \geq 4$) . Considering the first case we find that x_0 and y_0 cannot be irrational in (13) as irrational square roots do not give a rational number in sum.

Let us consider the second case when $n = \mu$. Indeed, from the one hand, there is Pythagorean triple of numbers z^m, x^m, y^m with $m = \mu/2$ such that $(z^m)^2 = (x^m)^2 + (y^m)^2$. On the other hand, the initial equality can be written in the form $z^2 = x_0^2 + y_0^2$ showing that the indicated triple of numbers corresponds to the triple z, x_0, y_0 describing the like right-angled triangle. Therefore $z^m/x^m = z/x_0$, $z^m/y^m = z/y_0$, $x^m = x_0 \cdot z^{m-1}$, $y^m = y_0 \cdot z^{m-1}$ and x_0 and y_0 are not irrational.

So, it was revealed, as a result of the previous calculation, that equality (13) consists of whole numbers. Furthermore, Fermat's triple obtained from them for the given $n > 2$, for example, x_0, y_0, x, is not the same by value as Fermat's triple x, y, z from (7), since $x_0 / y_0 \neq x / y$ that is clear from the following: $x_0^2/y_0^2 = x^n/y^n = (x^2/y^2)(x^{n-2}/y^{n-2})$.

Hence equality (13) followed from (12) describes another right-angled triangle different from that defined by Pythagorean triple x_0 , y_0 , z .

Let us come back to the assumption at the beginning of the proof that integer solution (7) exists. This assumption is substantiated only when there is a concrete solution (13) in whole numbers. In order to check validity of (13) it is necessary to do the same discourse as before, since equations (7) and (13) are identical by their properties. This procedure can be continued to infinity in the direction of decreasing whole numbers under condition that sequence of different chained equalities never stops and numbers x_0^2 and y_0^2 in (12) will be always whole. If it is not so, i.e., x_0^2 and y_0^2 in chained equalities (13) turn out to be fractions, then this means that solution (7) does not exist in the system of right-angled numbers**.** Actually, since all partitions of the type (12) are built from the very beginning exclusively on the set of right-angled numbers' squares being in fact whole items of finite series of partitions, then non-whole x_0^2 and y_0^2 show pointlessness of such procedure, i.e., the absence of integer solution (7) or zero solution. On the other hand, infinite sequence of chained equalities (13) leads to infinite decreasing of positive whole numbers that is impossible and therefore assuming that there exists an integer solution of (7) when $n > 2$ is not true. Thus the theorem is proved both for all even and for all odd powers of whole numbers.

2.3. Results and discussion.

So, the full proof of Beal's Conjecture is obtained owing to Fermat's method of infinite descent (see above). Materials and methods of this research are described in detail in the previous sections of the article. One should draw special attention to constructing chains of proportions (8), which lead to the basic equality of partitions (12). This equality is founded on the one-to-one correspondence described by *Lemma* and establishes isomorphism between partitions in two terms and partitions in three terms for each pair of whole numbers x_0 , y_0 . Indeed, partitions (12) are simultaneously one and the same partition in two terms similar to that defined by Pythagorean equation $z^2 = x_0^2 + y_0^2$. It should be noticed that these partitions have specific geometrical view being partitions of *n*-dimensional cubes into smaller *n*-dimensional cubes, so they are not linear sections of segments representing whole powers z^n, x_0^n, y_0^n, x^n, y^n on a line with improper topological structure. Thus partitions in (12) are equal similar partitions of *n*-dimensional cubes when one of the parts of the two-termed partition is divided in two parts with help of x_0 , y_0 in *n*-dimensional arithmetical space.

3. Conclusion.

Returning to the full proof of Beal's Conjecture, let us note the important circumstance, allowing to complete it, consisted in the true proof of Fermat's Last

Theorem as particular case of the considered conjecture. So, Beal's Conjecture led to solving the centuries-old problem of mankind and can be called rightfully Generalized Fermat's Last Theorem in memory of the greatest discovery in the history of human science opening new ways in world cognition and understanding ancient knowledge by means of modern mathematical language using the notion of isomorphism of mathematical objects and the reverse problem method of applied mathematics [3;4;5]. Particularly, solution of Beal's Conjecture contains in itself the description of a new hypothetical mathematical object with simple properties conditioned only by its intrinsic structure. Fig. 1 from [3] can give some visual impression about this structure when constructing similar right angle triangles from Pythagorean triple z , x_0 , y_0 .

This object (represented by hypothetical Fermat's equality in whole numbers) has been unknown in pure mathematics till now and can be understood on the following scheme (Fig. 3) of reverse problem solving showing identical transformations of one and the same partition $z^n = x^n + y^n$:

$$
\begin{array}{ccc}
z^n = x^n + y^n & \leftarrow & z^n = (x_0^{\,n} + y_0^{\,n}) + \lambda_n \\
\downarrow & & \uparrow \\
z^n = z^{n-2}(x_0^{\,2} + y_0^{\,2}) & \rightarrow & z^n = x_0^{\,n} + y_0^{\,n} + \lambda_n
\end{array}
$$

Fig. 3 (see designations in the text)

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*Статья опубликована в журнале Агентства Международных Исследований «Новая наука: от идеи к результату». Итоги Международной научно-практической конференции 29 февраля 2016 г. В 3-х частях. Ч. 3 – Стерлитамак: РИЦ АМИ, 2016 – 160 с. С. 36-42.