Mathematical Solution of Beal's Conjecture *

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Abstract: The Beal conjecture is proved by arithmetic geometry methods known yet to ancient mathematicians and developed by the author. These methods include constructing powers of whole numbers by means of proportions, making up partitions from them, their scaling-up and scaling-down in order to get equal similar partitions. As a result of such transformations, the Beal equation comes to the Fermat equation, which has no solution in positive whole numbers that is proved by the same methods plus Fermat's method of infinite descent. The given research is fulfilled in the system of right-angled numbers introduced by the author and leading to the mathematical discovery of Beal's Conjecture solution.

Keywords: Beal's Conjecture solution, Fermat's Last Theorem, arithmetic geometry, partitions, ancient mathematics.

AMS Subject Classification: 11 Number theory, 11G Arithmetic algebraic geometry (11G99), 11P Additive number theory; partitions (11P99).

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1. Introduction. Beal's Conjecture as generalized Fermat's Last Theorem.

Beal's Conjecture [1] deals with arbitrary powers of whole numbers combined in one equation similarly to the well-known equation of Fermat's Last Theorem. The Beal proposition can be solved by the ancient Greek arithmetic geometry methods applied successfully as well to the Fermat problem [2]. Among all well-known mathematics conjectures Beal's Conjecture is occupying a peculiar place being a generalization of Fermat's Last Theorem [1]. However the generalization in [1] concerns only the formal record of this conjecture and does not summarize the methods of proving Fermat's Last Theorem. On the contrary, the Beal conjecture comes to the Fermat problem considered as an arithmetic geometry problem with elements of combinatorics and has easy simple solution obtained by additive number theory methods apparently available to ancient mathematicians and Fermat too [2]. The given proof of Beal's Conjecture can be related to the part of number theory defined as arithmetic algebraic geometry.

2. New arithmetic geometry of Beal's Conjecture and Fermat's Last Theorem (solution of both).

The Beal conjecture states [1]:

The equation $A^x + B^y = C^z$ has no solution in positive integers A, B, C, x, y, and z with x, y, and z at least 3 and A,B, and C coprime.

Or, restated [1]:

Let A, B, C, x, y, and z be positive integers with x, y, z > 2. If $A^x + B^y = C^z$, then A, B, and C have a common factor.

Let us rewrite the Beal conjecture equality in the following way:

$$x^n + y^n = z^n \qquad (1)$$

with positive integers x, y, z having a common factor and exponent n taking simultaneously the next spectrum of values: $n = (k, l, m)$, where integers k, l, m

at least 3 and n has one independent value for each term. Thus we assume at the beginning that equality (1) exists, or partitions of the type (1) can be obtained. This method of proof is related to plausible reasoning and called the rule of contraries. Then one can explore some arbitrary solutions of equation (1) in whole numbers.

Consider equality (1) as a partition of whole number z^n into two whole parts x^n and y^n. It resembles the Pythagorean equation in real numbers, if we could reduce powers in (1) to the degree 2 with whole parts in the similar partition in order to carry further investigation. To produce such scaling, let us introduce the notion of right-angled numbers (these numbers are different from so called right angle triangle numbers representing Pythagorean triples).

Definition. Right-angled number is such a non-negative real number, the square of which is a whole non-negative number.

The set of right-angled numbers $\mathbf{P} = \{0, 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots\}$ is countable. The system of right-angled numbers $P = \langle \mathbf{P},+,\cdot,0,1\rangle$ is defined by operations of addition and multiplication and two singled out elements (zero and unit). The system P is non-closed in relation to addition. Notice that the set of non-negative whole numbers is a subset of the set of right-angled numbers. Then consider (1) on the 2-dimensional lattice of right-angled numbers with coordinates x_o , y_o and that which we call the norms of right-angled number z differing from each other by the value of its summands: $z^2 = x_o^2 + y_o^2$. The norm of non-zero right-angled numbers is always whole and cannot be less than 1. Whole numbers x_o^2 and y_o^2 run through values from 1 to z^2 and from z^2 to 1 one by one. So number z has z^2 different partitions as its norms.

For the purpose of reducing (1) to the view of Pythagorean equation in the system of right-angled numbers, one can rewrite (1) as an equality for some coprime

x', y', z', and common whole factor d : $(x'd)^k + (y'd)^l = (z'd)^m$ and fulfill scaling-down:

$$(z'd)^2 = (x'd)^k / (z'd)^{m-2} + (y'd)^l / (z'd)^{m-2} = (x')^k\, d^{k-m+2} / (z')^{m-2} + (y')^l\, d^{l-m+2} / (z')^{m-2} =$$

$x_o^2 + y_o^2$, where x_o^2 and y_o^2 with appropriate d are squares of some right-angled numbers x_o and y_o. In other words, we seek such d that satisfies the above stated condition. To get whole parts in the sum of this equality, one must regard exponents $(k-m+2)$ and $(l-m+2)$ with base d tuple to $(z')^{m-2}$. Obviously, k and l have to be more or equal $m-1$. If k or l does not satisfy this rule, then equality (1) cannot be represented on the lattice of right-angled numbers and consequently constructed from natural numbers on this lattice. However, if $(k, l) \geq m-1$, equality (1) assumes the following character (quantic) after fulfilling scaling-up:

$$z^m = x^k + y^l = z^{m-2}\,(x_o^2 + y_o^2) \qquad (2)$$

Let us apply now the ancient method of making powers using Euclid's geometrical theorem [2] and produce two chains of proportions connected with each other with some equality presenting integer z as a sum of two whole numbers:

$$z/x_o = x_o/k = k/k_1 = \ldots = k_{m-3}/k_{m-2} \qquad (3)$$

$$z/y_o = y_o/l = l/l_1 = \ldots = l_{m-3}/l_{m-2}$$

where z, x_o , y_o are right-angled numbers from (2), m natural index at least 3,and $z = k + l$; k and l are some whole parts of z taken from the method of scaling-down (see below).

From proportions (3) one can obtain the next formulae:

$$x_o^2 = kz = (k_1 z / x_o)z\ ,\ \ x_o^3 = k_1 z^2 = (k_2 z / x_o) z^2\ , \ldots ,\ x_o^m = k_{m-2}\, z^{m-1}, \qquad (4)$$

$$y_o^2 = lz = (l_1 z / y_o)z\ ,\ \ y_o^3 = l_1 z^2 = (l_2 z / y_o) z^2\ , \ldots ,\ y_o^m = l_{m-2}\, z^{m-1},$$

where integers k and l are found from the basic equality (1):

$$z = (z'd) = (x'd)^k / (z'd)^{m-1} + (y'd)^l / (z'd)^{m-1} = k + l$$

and exponents k and l have to be more or equal m, if numbers k and l are whole with $d = (z')^{m-1}$ as a minimum (d can be some whole number divisible by this minimum).

From (2) and (4) we obtain equal similar partitions of z^n into two whole parts:

$$z^m = x^k + y^l = z^{m-2}(x_o^2 + y_o^2) = x^m + y^m, \tag{5}$$

where $x^k = (x^{k/m})^m = x^m$, $y^l = (y^{l/m})^m = y^m$ with whole x, y by construction (for simplicity we do not change here the designations for x, y although exponents k and l are tuple to m). Square roots of x^m, y^m are mean proportionals between x_o^2 and z^{m-2}, y_o^2 and z^{m-2} describing a bigger right triangle similar to that with sides x_o , y_o , z represented by x_o^2, y_o^2, z^2.

This gives that (1) comes to the Fermat equality in right-angled numbers:

$$x^m + y^m = z^m, \ m \geq 3 \tag{6}$$

with whole $x = x'd$, $y = y'd$, $z = z'd$, and d as some whole factor, in particular prime factor. One can prove Fermat's Last Theorem now with the same methods as above in order to fulfill solution of the Beal conjecture in full and one measure.

Let us rewrite Fermat's Last Theorem in its usual view:

$$z^n = x^n + y^n, \ n > 2 \tag{7}$$

Suppose that at least one solution was found. Then we shall try to construct such a solution and make certain of its impossibility. We shall work in the system of right-angled numbers (see above *Definition*).

Consider (7) on the 2-dimensional lattice of right-angled numbers with right-angled coordinates x_0 , y_0 and corresponding norm $z^2 = x_0^2 + y_0^2$ differing by its square fragments relating to definite right-angled coordinates and being a partition of number z^2 into two summands represented by non-negative whole numbers. The minimal (non-zero) norm (standard) of right-angled numbers equals 1.

To construct powers of whole numbers presented in (7), let us produce two chains of continued proportions connected with each other by the norm $z^2 = x_0^2 + y_0^2$:

$$z/x_0 = x_0/k = k/k_1 = \ldots = k_{n-3}/k_{n-2}$$

$$z/y_0 = y_0/l = l/l_1 = \ldots = l_{n-3}/l_{n-2} \tag{8}$$

where natural indices of the last terms of each chain in (8) are obtained from $n > 2$. Continued proportions (8) yield the following formulae:

$$kz = x_0^2,\ k_1z = x_0k,\ k_2z = x_0k_1,\ \ldots,\ k_{n-2}z = x_0k_{n-3}$$

$$lz = y_0^2,\ l_1z = y_0l,\ l_2z = y_0l_1,\ \ldots,\ l_{n-2}z = y_0l_{n-3} \tag{9}$$

$$x_0^2 = kz = (k_1z/x_0)z,\quad x_0^3 = k_1z^2 = (k_2z/x_0)z^2,\ \ldots,\ x_0^n = k_{n-2}z^{n-1}$$

$$y_0^2 = lz = (l_1z/y_0)z,\quad y_0^3 = l_1z^2 = (l_2z/y_0)z^2,\ \ldots,\ y_0^n = l_{n-2}z^{n-1} \tag{10}$$

It is necessary now to fix the norm for the partition of z^n into two like powers in (7). As in the case of Beal's Conjecture, let us assume that z, x, y in presupposed equality (7) have a common factor d, i. e., $z = (z'd)$, $x = (x'd)$, $y = (y'd)$, where z', x'. y' coprime. Thereupon divide equality (7) by z^{n-1} and get:

$z = (z'd) = (x'd)^n /(z'd)^{n-1} + (y'd)^n /(z'd)^{n-1} = k + l$, where k and l integers with $d = (z')^{n-1}$ as a minimum. From this and (9)-(10) it follows that $z^2 = x_0^2 + y_0^2$ and $z^n = z^{n-2} (x_0^2 + y_0^2)$ is a scaled-up modification of the norm $z^2 = x_0^2 + y_0^2$.

Further, one can obtain a singular partition of z^n into three terms from (10) for the given norm when $n > 2$:

$$z^n = x_0^n + y_0^n + \lambda_n \tag{11}$$

where $\lambda_n = z^{n-1} [(k - k_{n-2}) + (l - l_{n-2})]$ is a remainder after subtracting x_0^n and y_0^n out of z^n such that $\lambda_n > 0$ when $n > 2$ and $x_0 y_0 \neq 0$, $\lambda_n = 0$ when $n = 2$ and $x_0 y_0 \neq 0$,

$x_0 , y_0, \in [0, z]$, $z \in (0, \infty)$.

There exists one-to-one correspondence between each pair of numbers (x_0 , y_0) with norm $z^2 = x_0^2 + y_0^2$ from 2-dimensional arithmetic space and each corresponding partition of any whole power $n > 2$ of integer z from n-dimensional arithmetic space into the sum of the same powers of numbers x_0 , y_0 and remainder λ_n from (11). Isomorphism between the set of points of 2-dimensional Euclidean space with position vector length z and coordinates x_0 , y_0 , the set of partitions of z^2 into squares, and the sets of partitions (11) for any whole $n > 2$ can be written as follows : $\{z \Rightarrow (xo , yo)\} \leftrightarrow \{ z^2 = x_0^2 + y_0^2 \} \leftrightarrow \{ z^n = x_0^n + y_0^n + \lambda_n\}$,

where sets are generated by the next power similarities:

$z \leftrightarrow z^2 \leftrightarrow z^n$, $x_0 \leftrightarrow x_0^2 \leftrightarrow x_0^n$, $y_0 \leftrightarrow y_0^2 \leftrightarrow y_0^n$, and is proved by (8) – (11).

Partitions (11) can be reduced to the norm, from which they were obtained:

$$z^n = x_0^n + y_0^n + \lambda_n = z^{n-2} (x_0^2 + y_0^2) = x^n + y^n \tag{12}$$

Formula (12) represents by itself a combinatorial equality of two partitions in three and two terms because there is one-to-one correspondence between pairs (x_0, y_0) and presupposed partition (7). In the case of right-angled numbers this equality is realized only when x_0, y_0 integers. Algorithm of such correspondence is given in the next formula (13). Thus scaling invariance of the norm $z^2 = (x_0^2 + y_0^2)$ leads to the following equalities of different fragments of partitions (12):

$$x_0^n + y_0^n = (x^n \text{ or } y^n) \qquad (13)$$

and correspondingly $\lambda_n = (y^n \text{ or } x^n)$. It can be noticed that $x_0^n \neq z^{n-2} \cdot y_0^2 = y^n$ and $y_0^n \neq z^{n-2} \cdot x_0^2 = x^n$ because of the lack of coincidence of decompositions in factorization of numbers x_0^n and y^n, y_0^n and x^n. Obviously, $x_0^n \neq z^{n-2} \cdot x_0^2$ and $y_0^n \neq z^{n-2} \cdot y_0^2$.

Let us show now that x_0 and y_0 cannot be irrational in (13) on account of integer partition of z^n into x^n and y^n. Here two cases can occur: when n is an odd number (designate it by $\nu = n_{\mathrm{odd}} \geq 3$) and when n is an even number (designate it by $\mu = n_{\mathrm{even}} \geq 4$). Considering the first case we find that x_0 and y_0 cannot be irrational in (13) as irrational square roots do not give a rational number in sum.

Let us consider the second case when $n = \mu$. Indeed, from the one hand, there is Pythagorean triple of numbers z^m, x^m, y^m with $m = \mu/2$ such that $(z^m)^2 = (x^m)^2 + (y^m)^2$. On the other hand, the initial equality can be written in the form $z^2 = x_0^2 + y_0^2$ showing that the indicated triple of numbers corresponds to the triple z, x_0, y_0 describing the like right-angled triangle. Therefore $z^m/x^m = z/x_0$, $z^m/y^m = z/y_0$, $x^m = x_0 \cdot z^{m-1}$, $y^m = y_0 \cdot z^{m-1}$ and x_0 and y_0 are not irrational.

So, it was revealed, as a result of the previous calculation, that equality (13) consists of whole numbers. Furthermore, Fermat's triple obtained from them for the given $n > 2$, for example, x_0, y_0, x, is not the same by value as Fermat's triple x, y, z from (7), since $x_0 / y_0 \neq x / y$ that is clear from the following: $x_0^2/y_0^2 = x^n/y^n = (x^2/y^2)(x^{n-2}/y^{n-2})$.

Hence equality (13) represented in the form (12) describes another right-angled triangle different from that defined by Pythagorean triple x_0 , y_0 , z .

Let us come back to the assumption at the beginning of the proof that integer solution (7) exists. This assumption is substantiated only when there is a concrete solution (13) in whole numbers. In order to check validity of (13) it is necessary to do the same discourse as before, since equations (7) and (13) are identical by their properties. This procedure can be continued to infinity in the direction of decreasing whole numbers under condition that sequence of different chained equalities never stops and numbers x_0^2 and y_0^2 in (12) will be always whole. If it is not so, i.e., x_0^2 and y_0^2 in chained equalities (13) turn out to be fractions, then this means that solution (7) does not exist in the system of right-angled numbers**.** Actually, since all partitions of the type (12) are built from the very beginning exclusively on the set of right-angled numbers' squares being in fact whole items of finite series of partitions, then non-whole x_0^2 and y_0^2 show pointlessness of such procedure, i.e., the absence of integer solution (7) or zero solution. On the other hand, infinite sequence of chained equalities (13) leads to infinite decreasing of positive whole numbers that is impossible and therefore assuming that there exists an integer solution of (7) when $n > 2$ is not true. Thus the theorem is proved both for all even and for all odd degrees of whole numbers and for any finite whole x, y, z, d.

3. Conclusion.

Returning to the full proof of Beal's conjecture, let us note the important circumstance, making possible to complete it, consisted in the true proof of Fermat's Last Theorem as particular case of the considered conjecture. So, Beal's Conjecture led to solving the centuries-old problem of mankind and can be called rightfully Generalized Fermat's Last Theorem in memory of the greatest discovery in the history of human science opening new ways in world cognition and understanding ancient knowledge by means of modern mathematical language [3;4;5].

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